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Reducibility and nonbinding

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Abstract

This paper gives equivalent descriptions of reducible and weakly reducible Heegaard splittings in terms of nonbinding.

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0. Introduction

A Heegaard splitting (V, V') of a closed orientable 3-manifold M is said to be reducible if there are essential disks D and D' properly embedded in V and V' respectively so that $\partial D = \partial D'$, and is said to be weakly reducible if there are essential disks D and D' properly embedded in V and V' respectively so that ∂D and $\partial D'$ are disjoint in $\partial V = \partial V'$. It is easy to see that a reducible Heegaard splitting is weakly reducible.

A Heegaard splitting (V, V') is said to be nonbinding if there is a Heegaard diagram $(V; J_1, \ldots, J_n)$ associated to (V, V') so that $\Gamma = \{J_1, \ldots, J_n\}$ does not bind the free group $F_n = \pi_1(V)$, and is said to be weakly nonbinding if there is a Heegaard diagram $(V; J_1, \ldots, J_n)$ associated to (V, V') so that a nonempty subset of $\Gamma = \{J_1, \ldots, J_n\}$ does not bind the free group $F_n = \pi_1(V)$, and in particular, is said to be nearly nonbinding if the nonempty subset, which does not bind the free group $F_n = \pi_1(V)$, of $\Gamma = \{J_1, \ldots, J_n\}$ has exactly $n - 1$ curves. It is easy to see that a nonbinding Heegaard splitting is weakly nonbinding and a nearly nonbinding one is weakly nonbinding.

In this paper we will prove that a Heegaard splitting is reducible if and only if it is nonbinding and a Heegaard splitting is weakly reducible if and only if it is weakly nonbinding. Moreover, we get a partial converse of the above result which says that

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a nearly nonbinding Heegaard splitting is nonbinding (therefore reducible). In fact, we obtain a more general result.

1. Preliminaries

We work in the piecewise linear category and all 3-manifolds are assumed to be orientable and, except for handlebodies, to be closed as well in this paper.

A 2-sphere in a 3-manifold M is essential if it does not bound a 3-ball in M. A 3-manifold M is reducible if it contains an essential 2-sphere.

Let S be a surface in a 3-manifold M which is either properly embedded or contained in ∂M. An essential disk in (M, S) is a disk D in M such that $D \cap S = \partial D$ and ∂D is essential in S. If such a disk exists, we say that S is compressible in M; otherwise, it is incompressible.

A handlebody H is the boundary connected sums of a finite number of copies of $S^1 \times D^2$. A properly embedded disk D in H is called a meridian disk of H if the manifold obtained by cutting H along D is still a handlebody, and a collection Γ of pairwise disjoint n meridians, $D_1, \ldots, D_n$, in H is called a complete system of H if the manifold obtained by cutting H open along Γ is a 3-ball. n is called the genus of H. Furthermore, a collection of pairwise disjoint simple closed curves (s.c.c.) on the boundary of H is called a complete system of H (or ∂H) if it bounds a complete system of meridian disks of H.

Let M be a 3-manifold. A Heegaard splitting (or H-S) (V, V') of M is a representation of M as $V \cup_S V'$, where V and V' are homeomorphic handlebodies of some fixed genus n and $V \cap V' = \partial V = \partial V' = S$, a Heegaard surface. Let $\Gamma = \{J_1, \ldots, J_n\} \subset S$ be a complete system of V', then $(V; \Gamma)$ is called a Heegaard diagram (or H-D) of M associated to the H-S (V, V').

Definition 1.1. Let (V, V') be an H-S of a 3-manifold M. If there are essential disks D and D' properly embedded in V and V' respectively so that $\partial D = \partial D'$, we say that (V, V') is reducible; if there are essential disks D and D' properly embedded in V and V' respectively so that ∂D and $\partial D'$ are disjoint in the Heegaard surface S, we say that (V, V') is weakly reducible.

It is easy to see that a reducible H-S is weakly reducible. It is a theorem of Casson and Gordon [1] that if (V, V') is a weakly reducible H-S of M then either M contains an incompressible surface, or (V, V') is reducible. It is a theorem of Haken [2] that any H-S of a reducible 3-manifold is reducible.

Definition 1.2. Let $W \subset F_n$ be a set of cyclically reduced words in the free group F_n with a basis X. The incidence graph $J(W)$ is the graph whose vertices are in 1–1 correspondence with the nontrivial words in W, with an edge joining vertices w_1 and w_2 if there exists $x \in X$ such that x or x^{-1} lies in w_1 and x or x^{-1} lies in w_2. W is connected with respect to the basis X if $J(W)$ is connected, and is connected if it is connected with respect to each basis of F_n. If the set W of cyclic elements is not contained in any proper free factor of F_n and if W is connected, we say that W binds F_n.

Let $\Gamma = \{J_1, \dots, J_m\}$ be a collection of pairwise disjoint s.c.c. on the boundary of a handlebody H of genus n, with each J_i essential on ∂H. We will abuse the notation Γ slightly to represent the corresponding cyclically reduced elements of $\pi_1(H) = F_n$, a free group of rank n, whenever an orientation of Γ is given, if this causes no confusion. The following lemma is an immediate consequence of [5, Corollary 1].

Lemma 1.3. *$\partial H - \Gamma$ is incompressible in H if and only if Γ binds the free group $\pi_1(H) = F_n$.*

Definition 1.4. An H-D $(V; J_1, \dots, J_n)$ is called nonbinding, weakly nonbinding or nearly nonbinding if $\Gamma = \{J_1, \dots, J_n\}$, a nonempty subset of Γ or a $(n-1)$-subset (when $n \geqslant 2$) of Γ does not bind the free group $\pi_1(V) = F_n$ respectively. An H-S (V, V') is called nonbinding, weakly nonbinding or nearly nonbinding respectively if (V, V') has an associated H-D which is nonbinding, weakly nonbinding or nearly nonbinding respectively.

It is easy to see that a nonbinding H-S is weakly nonbinding and a nearly nonbinding H-S is weakly nonbinding.

Let J_1, J_2 be two nontrivial s.c.c. on a connected surface S, β a simple arc on S connecting J_1 and J_2 with $\beta \cap (J_1 \cup J_2) = \partial\beta$. A regular neighborhood of $\beta \cup (J_1 \cup J_2)$ on S has three boundary components. The one which is not isotopic to neither J_1 nor J_2 is called a band connected sum of J_1 and J_2 along β. Let $\Gamma, \Gamma' \subset \partial V$ be two complete systems of handlebody V. It is known that Γ' can be obtained from Γ by operating a finite number of band connected sums on Γ and vice versa.

A surface is said to be planar if it is compact, connected and embeds in R^2.

2. The equivalence between reducibility and nonbinding

First we prove the equivalence between reducibility and nonbinding for an H-S.

Theorem 2.1. *An H-S is nonbinding if and only if it is reducible.*

Proof. Assume (V, V') is a nonbinding H-S, then there exists an associated H-D $(V; \Gamma)$ such that Γ does not bind the free group $\pi_1(V) = F_n$. By Lemma 1.3, $\partial V - \Gamma$ is compressible in V. It is easy to see that for any compressing disk D of $\partial V - \Gamma$ in V, ∂D bounds an essential disk in V'. So (V, V') is reducible.

Now we assume (V, V') is reducible, then there are essential disks D and D' in V and V' respectively with $\partial D = \partial D'$. It is easy to find a complete system $\Gamma \subset \partial V'$ of V' with D as a compressing disk of $\partial V - \Gamma$ in V. So (V, V') is nonbinding.

The following theorem sets up the equivalence between weak nonbinding and weak reducibility for an H-S.

Theorem 2.2. *An H-S is weakly nonbinding if and only if it is weakly reducible.*

Proof. Let (V, V') be a weakly nonbinding H-S. Then there exists an H-D $(V; J_1, \ldots, J_n)$ associated to (V, V') such that a nonempty subset Γ' of $\Gamma = \{J_1, \ldots, J_n\}$ does not bind the free group $\pi_1(V) = F_n$. By Lemma 1.3, $\partial V - \Gamma'$ is compressible in V. For any compressing disk D of $\partial V - \Gamma'$ in V, ∂D is essential on ∂V and ∂D is disjoint from any curves in Γ' which are meridian disks of V', so (V, V') is weakly reducible.

For the other direction, assume that an H-S (V, V') is weakly reducible, then there are essential disks D and D' properly embedded in V and V' respectively with $\partial D \cap \partial D' = \emptyset$. In case the genus $g(V)$ is 1, the conclusion certainly holds. Assume $g(V) \geqslant 2$.

If D separates V, D cuts V into two handlebodies V_1 and V_2 with $g(V_i) \geqslant 1$, $i = 1, 2$. Say $\partial D' \subset \partial V_1 - D$. If $\partial D'$ does not separate ∂V_1, $\partial D'$ can be extended to a complete system of V', and $\partial D' \in \pi_1(V_1)$, a proper free factor of $\pi_1(V)$. If $\partial D'$ separates ∂V_1, D' separates V' into two handlebodies V_1' and V_2', say $\partial D \subset \partial V_1'$, then $g(V_2') \geqslant 1$, and we can choose a meridian disk D'' of V_2' with $\partial D'' \subset \partial V_2' - D \subset \partial V'$ which can be extended to a complete system of V' such that $\partial D''$ is contained in the proper free factor $\pi_1(V_2)$ of $\pi_1(V) = F_n$. So it is weakly nonbinding.

If D does not separates V, we can choose another essential disk D_1 properly embedded in V which separates V with $\partial D_1 \cap \partial D' = \emptyset$. Therefore we can follow the above steps to finish the proof.

Connecting with Casson and Gordon's result [1], we can get

Corollary 2.3. *Assume an H-S (V, V') of M is weakly nonbinding, then either (V, V') is reducible or M contains an incompressible surface.*

Clearly, an H-S with genus 1 is nonbinding (or reducible) if and only if it is weakly nonbinding (or weakly reducible). It is easy to find examples which indicate that reducibility is stronger than weak reducibility. However, the following theorem gives a sufficient condition for a weak nonbinding (or weak reducible) H-S to be nonbinding (or reducible).

Theorem 2.4. *Let M be a 3-manifold. Assume (V, V') is a nearly nonbinding H-S of M. Then (V, V') is nonbinding.*

Proof. By assumption, there is an H-D $(V; J_1, \ldots, J_n)$ associated to (V, V'), where $n \geqslant 2$, such that a $(n-1)$-subset Γ' of $\Gamma = \{J_1, \ldots, J_n\}$ does not bind the free group $\pi_1(V) = F_n$. Say, $\Gamma' = \{J_1, \ldots, J_{n-1}\}$. If Γ does not bind F_n, (V, V') is nonbinding. In the following we assume that Γ binds F_n, or equivalently, $\partial V - \Gamma$ is incompressible in V. Now by Lemma 1.3, $\partial V - \Gamma'$ is compressible in V. Let D be a compressing disk of $\partial V - \Gamma'$, then $\partial D \cap J_n \neq \emptyset$.

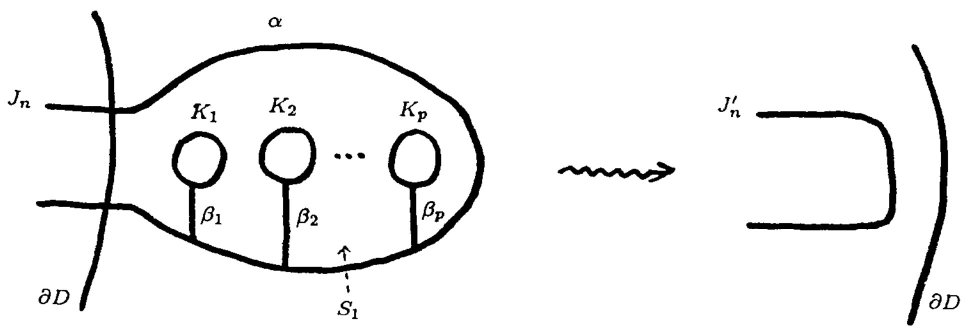

Fig. 1.

We first assume D separates V into two handlebodies V_1 and V_2 with $g_i = g(V_i) \geqslant 1$ for $i = 1, 2$. Since $\partial V - \Gamma'$ is a torus with $2n - 2$ holes, one of ∂V_1 and ∂V_2, say ∂V_1, contains exactly a g_1-subset Γ'' of Γ' and ∂V_2 contains a $(g_2 - 1)$-subset $\Gamma' - \Gamma''$ of Γ'. Say $\Gamma'' = \{J_1, \ldots, J_{g_1}\}$. Let S be the surface obtained by cutting open $\partial V_1 - \mathrm{Int}(D)$ along Γ'', then S is a planar surface and ∂S consists of ∂D and all the cutting sections of Γ''. Without losing generality we assume that J_n intersects ∂D transversely and $J_n \cap \partial D$ has minimal components up to isotopy. By $\partial D \cap J_n \neq \emptyset$ we know that each component of $J_n \cap \partial D$ is an essential simple arc with its two end points lying in ∂D and separates S into two planar surfaces in which each one has at least two boundary components. Choose a component α of $J_n \cap \partial D$ such that α cuts out a planar surface S_1 with $\mathrm{Int}(S_1) \cap J_n = \emptyset$. Assume $\partial S_1 = \{K_0, K_1, \ldots, K_p\}$, where K_0 contains α and $\{K_1, \ldots, K_p\}$ is a nonempty subset of the cutting section sets of Γ''. For each j, $1 \leqslant j \leqslant p$, choose a simple arc β_j in S_1 connecting α with K_j such that $\beta_{j_1} \cap \beta_{j_2} = \emptyset$ if $j_1 \neq j_2$. See Fig. 1. Then we make the band sums

$$J_n' = J_n \underset{\beta_1}{\#} K_1 \underset{\beta_2}{\#} \cdots \underset{\beta_p}{\#} K_p.$$

Thus, after an isotopy, $\{J_1, \ldots, J_{n-1}, J_n'\}$ is a complete system of V' and $J_n' \cap S$ has less components than $J_n \cap S$ has. After operating a finite number of such steps, we can get a complete system Γ^* of V', say $\Gamma^* = \{J_1, \ldots, J_{n-1}, J_n^*\}$, such that $\Gamma'' = \{J_1, \ldots, J_{g_1}\} \subset \partial V_1$ and $\Gamma^* - \Gamma'' = \{J_{g_1+1}, \ldots, J_{n-1}, J_n{}^*\} \subset \partial V_2 - D$. Thus D is a compressing disk of $\partial V - \Gamma^*$ in V and so Γ^* does not bind the free group $\pi_1(V) = F_n$. Hence $(V; \Gamma^*)$ is a nonbinding H-D associated to (V, V') and the H-S (V, V') is nonbinding.

We now assume that D does not separate V. Let F be the surface obtained by cutting ∂V along Γ'. If ∂D does not separate F, the surface F' obtained by cutting F along ∂D is a planar surface with $2n$ boundary components. Choose a s.c.c. J in $\mathrm{Int}(F')$ such that J divides F' into two planar surfaces, one of which contains only the two cutting sections of ∂D and the other one contains all the cutting sections of Γ'. Then J is a disk curve in both V and V'. So (V, V') is reducible, and is nonbinding. If ∂D separates

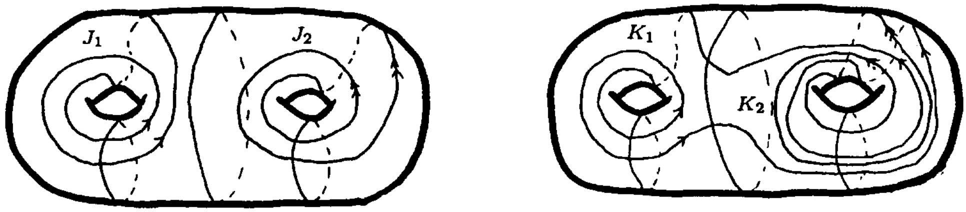

Fig. 2.

F, then ∂D cuts out a planar surface F'' from F. This means ∂D bounds a disk in V'. Hence (V, V') is reducible.

The following corollary is an immediately consequence of the above theorem.

Corollary 2.5. *An H-S of genus* 2 *is reducible if and only if it is weakly reducible.*

Let (V, V') be an H-S of a 3-manifold M. Assume (V, V') is weakly reducible. Then there is an associated H-D $(V; \Gamma = \{J_1, \ldots, J_n\})$ such that a nonempty subset Γ' of Γ does not bind the free group $\pi_1(V) = F_n$, or equivalently, $\partial V - \Gamma'$ is compressible in V. Let D be a compressing disk of $\partial V - \Gamma'$ in V. We can use the method similar to the proof of Theorem 2.4 to get the following general result.

Theorem 2.6. *A weakly reducible H-S* (V, V') *is reducible if the above* D *can be chosen in such a way that* D *cuts* V *into two handlebodies* V_1 *and* V_2 *with a nonempty p-subset* Γ'' *of* Γ' *entirely lying in* ∂V_1, *where* $p = g(V_1)$.

Remark 2.7. An H-S is nonbinding does not imply each H-D associated to it is nonbinding. See the following example (Fig. 2).

Consider a orientable handlebody H_2 of genus 2, with two systems of curves: J_1, J_2 and K_1, K_2.

In the natural presentation of $\pi_1(H_2)$, when H_2 is embedded in $\mathbb{R}^3$ as on the picture we have: $[J_1] = x_1^2$, $[J_2] = x_2^2$ and $\partial H_2 - J_1 - J_2$ is compressible in H_2, therefore $\{J_1, J_2\}$ does not bind $\pi_1(H_2)$.

In $\pi_1(H_2)$, curves K_1 and K_2 can be written: $[K_1] = x_1^2 x_2^2$, $[K_2] = x_2^2$ and since $\{K_1, K_2\}$ binds $\pi_1(H_2)$, $\partial H_2 - K_1 - K_2$ is incompressible in H_2.

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